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A COHERENCY MODEL FOR SPATIALLY VARYING GROUND MOTIONS

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SUMMARY

A theoretical model for the coherency function describing spatial variability of earthquake ground motions is developed. The model consists of three components characterizing three distinct effects of spatial variability, namely, the incoherence effect that arises from scattering of waves in the heterogeneous medium of the ground and their differential superpositioning when arriving from an extended source, the wave-passage effect that arises from difference in the arrival times of waves at different stations, and the site-response effect that arises from difference in the local soil conditions at different stations. Attenuation of waves, which also gives rise to spatial variability, is shown to have little influence on the coherency function. It is shown that the incoherence component of the coherency function is a real-valued, non-negative, decaying function of frequency and interstation distance, whereas the wave-passage and site-response components are complex functions of unit modulus that characterize the phasing of the wave components. A parametric study reveals that the site-response effect can be more significant for short- or medium-span structures situated in regions with rapidly varying local soil conditions, whereas the wave-passage effect can be more significant for long-span, flexible structures.

KEY WORDS: bridges; coherency; earthquakes; ground motions; stochastic processes

INTRODUCTION

Four distinct phenomena give rise to the spatial variability of earthquake-induced ground motions: (1) loss of coherency of seismic waves due to scattering in the heterogeneous medium of the ground, as well as due to the differential superpositioning of waves arriving from an extended source, collectively denoted herein as the 'incoherence' effect; (2) difference in the arrival times of waves at separate stations, denoted herein as the 'wave-passage' effect; (3) gradual decay of wave amplitudes with distance due to geometric spreading and energy dissipation in the ground medium, denoted herein as the 'attenuation' effect; and (4) spatially varying local soil profiles and the manner in which they influence the amplitude and frequency content of the bedrock motion underneath each station as it propagates upward, denoted herein as the 'site-response' effect. These effects are well characterized by the so called 'coherency function', which is the normalized cross-power spectral density of the motions at two stations.

Recent empirical studies with array recordings have shed light on the nature and magnitude of the incoherence and wave-passage effects (e.g. References 1–3). These are primarily based on recordings at the SMART-1 array in Taiwan, where soil conditions are more or less uniform throughout the array. These effects tend to increase with increasing distance between the stations, or with increasing frequency of the ground motion. Schneider *et al.*[4] have compared the estimated coherencies for ten dense arrays situated on different but uniform soil conditions. Their results indicate that coherencies computed for arrays located on alluvium are similar, and that coherencies for arrays located on rock are generally lower than those for arrays located on alluvium.

The attenuation effect is often insignificant for typical sizes of man-made structures. Furthermore, as shown later, while attenuation affects both the amplitude and frequency content of the waves, it has little influence on the coherency function.

The effect of site response depends on the variation of the local soil profile with distance and, therefore, is site specific. Furthermore, a characterization of this effect cannot be derived from recordings of an array that

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is situated on uniform soil. Somerville *et al.*[5] have suggested using dense array recordings of small earthquakes to determine the site-specific component of the coherency function, and then combine it with the coherency estimated by simulation of a design earthquake on a nearby extended source. Unfortunately, there are not many densely instrumented sites and this approach is severely limited. It is clear that for estimation of a site-specific coherency function reliance solely on an empirical approach is not practical.

Spatial variability of the strong ground motion can significantly influence internal forces induced in structures with multiple supports, such as bridges and viaducts. The variability in the support motions usually tends to reduce the inertia-generated forces within the structure, as compared to the forces generated in the same structure when the supports move uniformly. However, differential support motions generate additional forces, known as pseudo-static forces,[6] which are absent when the structure is subjected to uniform support motions. The resultants of the two sets of forces can exceed the level of forces generated in the structure with uniform support motions, particularly when the structure is stiff (e.g. see Reference 7).

Long-span structures, for which the incoherence and wave passage effects can be significant, are usually flexible. For these structures differential support motions often do not create a critical design situation, as the spatial variability effect usually tends to reduce the demand on the structure. On the other hand, differential support motions caused by the site-response effect could pose a serious problem for short-span, stiff structures that are situated in regions with rapidly varying soil profiles. This situation arises, for example, for several freeway viaducts in the San Francisco Bay Area. Hence, a correct characterization of the spatial variability of ground motion, and particularly the component due to the site-response effect, is essential for proper seismic design and retrofitting of multiply-supported structures.

In this paper, a theoretical model of the coherency function for one-dimensional ground motions and incorporating the incoherence, wave-passage, and site-response effects is developed. The model is derived from elementary notions of stationary random process theory and simplifying assumptions regarding the propagation of seismic waves. The model is in the form of the product of three terms representing the three distinct effects defined above. As mentioned earlier, the attenuation effect has little influence on the coherency function, and this is also shown. The components representing wave-passage and site-response effects, which only affect the phasing of the wave components, are expressed in terms of regional and local properties of the ground medium, i.e. the regional shear wave velocity and the properties of the local soil layers at each station. The component representing the incoherence effect requires an empirical determination.

Using a simple model to idealize soil profiles at two stations, a parametric study is performed to determine the relative importance of the wave-passage and site-response effects. The results indicate that the site-response effect can be more significant in a range of frequencies that is relevant to the response of short- and medium-span bridges and viaducts.

The coherency model developed in this paper together with the multiple-support response spectrum method developed earlier[7] provides a rational framework for seismic analysis of multiply supported structures with full account of the ground motion spatial variability effect.

THE COHERENCY FUNCTION

The coherency function characterizes the nature of the spatial variability of ground motion in the frequency domain. In the context of stationary random processes, it represents the cross-power spectral density of the motions at two stations, normalized by the square root of the corresponding auto-power spectral densities. For ground acceleration processes $a_k(t)$ and $a_l(t)$ at stations k and l, respectively, the coherency function is defined by

$$\gamma_{kl}(\omega) = \begin{cases} \dfrac{G_{a_k a_l}(\omega)}{\sqrt{G_{a_k a_k}(\omega) G_{a_l a_l}(\omega)}} & \text{for } G_{a_k a_k}(\omega) G_{a_l a_l}(\omega) \neq 0 \\ 0 & \text{for } G_{a_k a_k}(\omega) G_{a_l a_l}(\omega) = 0 \end{cases} \tag{1}$$

where ω denotes the circular frequency, $G_{xx}(\omega)$ denotes the auto-power spectral density of the process $x(t)$, and $G_{xy}(\omega)$ denotes the cross-power spectral density of the processes $x(t)$ and $y(t)$. In general, $\gamma_{kl}(\omega)$ is

complex valued. Its bounded modulus, $0 \leqslant |\gamma_{kl}(\omega)| \leqslant 1$, often called lagged coherency, is a measure of linear statistical dependence between the two processes. In particular, $\gamma_{kl}(\omega) = 1$ denotes perfect linear dependence between the two processes (e.g. identical and in-phase motions), whereas $\gamma_{kl}(\omega) = 0$ denotes complete lack of linear dependence, which is equivalent to statistical independence if the processes are Gaussian.

Equation (1) can be written in the form

$$\gamma_{kl}(\omega) = |\gamma_{kl}(\omega)| \exp[\mathrm{i}\theta_{kl}(\omega)] \tag{2}$$

where $\mathrm{i} = \sqrt{-1}$, and

$$\theta_{kl}(\omega) = \tan^{-1} \frac{\operatorname{Im} \gamma_{kl}(\omega)}{\operatorname{Re} \gamma_{kl}(\omega)} \tag{3}$$

is a frequency-dependent phase angle, where Im and Re refer to the imaginary and real parts of a complex function. It will be shown subsequently that the two terms in equation (2) characterize distinct effects of spatial variability: the real-valued function $|\gamma_{kl}(\omega)|$ characterizes the incoherence effect, whereas the complex term $\exp[\mathrm{i}\theta_{kl}(\omega)]$ characterizes the wave-passage and site-response effects. The latter term represents a frequency-dependent shift in the phasing of the two processes.

In this paper, we make use of a discrete representation of a process in terms of its frequency components. It is well known that a stationary process $a(t)$ can be approximately decomposed into a set of discrete frequency components in the form

$$a(t)^{\mathrm{d}} = \sum_{i=1}^{n} A_i \cos(\omega_i t + \phi_i) \tag{4}$$

where A_i, $i = 1, 2, \ldots, n$, are zero-mean, uncorrelated random amplitudes with mean squares $E[A_i^2] = \sigma_i^2$, $\omega_i = \omega_1 + (i-1)\Delta\omega$ are n equally spaced discrete frequency points at $\Delta\omega$ increments, and ϕ_i are random phase angles with uniform distributions over $(0, 2\pi)$ and statistically independent of each other and of the A_i's. The superposed 'd' refers to the discrete version of the process.

The auto-correlation function of the process in equation (4) is

$$\begin{aligned} E[a(t_1)a(t_2)]^{\mathrm{d}} &= \sum_{i=1}^{n} \sum_{j=1}^{n} E[A_i A_j \cos(\omega_i t_1 + \phi_i) \cos(\omega_j t_2 + \phi_j)] \\ &= \frac{1}{2} \sum_{i=1}^{n} \sigma_i^2 \cos[\omega_i (t_2 - t_1)] \end{aligned} \tag{5}$$

where $E[\cdot]$ denotes the expectation operator, and use has been made of the relation

$$\begin{aligned} E[\cos(\omega_i t_1 + \phi_i) \cos(\omega_j t_2 + \phi_j) &= \tfrac{1}{2} \cos[\omega_i (t_2 - t_1)] \quad \text{for } i = j \\ &= 0 \quad \text{for } i \neq j \end{aligned} \tag{6}$$

Taking the Fourier transform of equation (5), the power spectral density of the discretized process is obtained

$$G_{aa}(\omega)^{\mathrm{d}} = \frac{1}{4} \sum_{i=1}^{n} \sigma_i^2 [\delta(\omega_i - \omega) + \delta(\omega_i + \omega)] \tag{7}$$

where $\delta(\cdot)$ denotes the Dirac delta function. This function represents a series of spikes of amplitude $\sigma_i^2/4$ situated at discrete frequency points $\pm\omega_i$. For small $\Delta\omega$, this is essentially equivalent to a continuous-parameter power spectral density function $\tilde{G}_{aa}(\omega)$ defined by 'spreading' the spikes over the interval $\Delta\omega$ around each ω_i, i.e.

$$\tilde{G}_{aa}(\omega) = \frac{\sigma_i^2}{4\Delta\omega}, \quad \omega_i - \tfrac{1}{2}\Delta\omega \leqslant \omega < \omega_i + \tfrac{1}{2}\Delta\omega \tag{8}$$

It follows that as $\Delta\omega \to 0$ and $n \to \infty$, the discrete power spectral density $G_{aa}(\omega)^{\mathrm{d}}$ approaches the power spectral density $G_{aa}(\omega)$ of $a(t)$, provided

$$\sigma_i^2 = 4\Delta\omega G_{aa}(\omega_i) \tag{9}$$

This relation uniquely defines the mean squares of the random wave amplitudes A_i that are necessary for the discrete representation in equation (4). We note that, by virtue of the central limit theorem, the above representation also implies Gaussianity of the process.

In the following four sections, we employ the basic definition in equation (1) together with the above discrete representation of ground acceleration processes at stations k and l to develop a simple model for the coherency function that incorporates the four effects of incoherence, wave passage, attenuation and site response. In this analysis, the superscript d is dropped whenever a transition from the discrete to the continuous representation of a process or its auto-correlation or power spectral density functions is implied.

THE INCOHERENCE EFFECT

The incoherence effect is due to the scattering of waves in the heterogeneous medium of the ground as well as due to the differential superpositioning of waves arriving at each station from different segments of an extended source. Both these effects increase with increasing distance between the two stations. They are also expected to increase with the wave frequency, as high-frequency waves have short wavelengths and tend to be more affected by heterogeneities of the propagation path as well as by differential superpositioning. Due to the random nature of the fault rupture at the source and the heterogeneities in the ground medium, the incoherence effect can reasonably be described in a probabilistic sense.

Consider the discrete version of the acceleration process at station k,

$$a_k(t)^{\mathrm{d}} = \sum_{i=1}^{n} A_i \cos(\omega_i t + \phi_i) \tag{10}$$

The acceleration process at station l, a distance d_{kl} from station k, will have somewhat different wave amplitudes and phases. Due to the random nature of the incoherence effect, these differences in general will be random. To describe them, we write the discrete version of the acceleration process at station l as

$$a_l(t)^{\mathrm{d}} = \sum_{i=1}^{n} (p_{kl,i}A_i + q_{kl,i}B_i)\cos(\omega_i t + \phi_i + \varepsilon_{kl,i}) \tag{11}$$

where B_i are zero-mean, uncorrelated random variables with mean squares σ_i^2 representing the incoherent portion of the amplitudes, $\varepsilon_{kl,i}$ are independent random phase differences with zero mean and variances $\alpha_{kl,i}^2$, and $p_{kl,i}$ and $q_{kl,i}$ are deterministic coefficients defined in the interval (0, 1). Since B_i and $\varepsilon_{kl,i}$ represent random differences, it is natural to assume them to be statistically independent of A_i and ϕ_i. The parameters $p_{kl,i}$, $q_{kl,i}$ and $\alpha_{kl,i}$ are introduced to control the extent of incoherence between the two motions. As the subscript kl,i signifies, these parameters are dependent on the frequency ω_i and distance d_{kl} between the two stations. Based on the trends mentioned earlier, it should be clear that $p_{kl,i}$ is a decreasing function of the frequency and distance, whereas $q_{kl,i}$ and $\alpha_{kl,i}$ are increasing functions of these variables.

In the absence of wave-passage, attenuation, and site-response effects, the processes $a_k(t)$ and $a_l(t)$ must be statistically equivalent, i.e. they must have identical power spectral densities. This requires that the component phase angles of $a_l(t)^{\mathrm{d}}$ be uniformly distributed over the interval $(0, 2\pi)$, and that the mean squares of the wave amplitudes of $a_l(t)^{\mathrm{d}}$ be equal to σ_i^2. It is shown in Appendix I that the phase angles $\phi_i + \varepsilon_{kl,i}$, once wrapped within the interval $(0, 2\pi)$, are indeed uniformly distributed. The equivalence of the mean-square wave amplitudes is assured if we require that

$$E[(p_{kl,i}A_i + q_{kl,i}B_i)^2] = (p_{kl,i}^2 + q_{kl,i}^2)\sigma_i^2 = \sigma_i^2 \tag{12}$$

which imposes the condition

$$p_{kl,i}^2 + q_{kl,i}^2 = 1 \tag{13}$$

A simple way to satisfy this requirement is to set $p_{kl,i} \equiv \cos\beta_{kl,i}$ and $q_{kl,i} \equiv \sin\beta_{kl,i}$, where $\beta_{kl,i}$ is an acute angle dependent on the frequency and the inter-station distance. We also assume that the random phase differences $\varepsilon_{kl,i}$ are normally distributed. This assumption is justified by virtue of the central limit theorem, since these phase differences are the result of accumulation of many random scattering and superposition effects.

As described above, the angles $\alpha_{kl,i}$ and $\beta_{kl,i}$ are functions of the inter-station distance d_{kl} and the wave frequency ω_i. In a continuous form for all frequencies, we write them as $\alpha(d_{kl},\omega)$ and $\beta(d_{kl},\omega)$. As $d_{kl} \to 0$, the two processes $a_k(t)$ and $a_l(t)$ become identical and one must have

$$\lim_{d_{kl}\to 0} \alpha(d_{kl},\omega) = \lim_{d_{kl}\to 0} \beta(d_{kl},\omega) = 0 \tag{14}$$

Also, $\omega = 0$ implies an infinite wave length and a rigid-body motion of the ground, for which there must be perfect coherence. Hence,

$$\lim_{\omega\to 0} \alpha(d_{kl},\omega) = \lim_{\omega\to 0} \beta(d_{kl},\omega) = 0 \tag{15}$$

On the other hand, as $d_{kl} \to \infty$ or $\omega \to \infty$, we must have perfect incoherence between the two motions. Hence,

$$\lim_{d_{kl}\to \infty} \alpha(d_{kl},\omega) = \lim_{\omega\to \infty} \alpha(d_{kl},\omega) = \infty \tag{16}$$

$$\lim_{d_{kl}\to \infty} \beta(d_{kl},\omega) = \lim_{\omega\to \infty} \beta(d_{kl},\omega) = \frac{\pi}{2} \tag{17}$$

From the above analysis it is clear that $\alpha(d_{kl},\omega)$ and $\beta(d_{kl},\omega)$ are increasing functions of d_{kl} and ω within the limits described above.

Taking the expectation of the product of the processes in equations (10) and (11), the cross-correlation function is found to be

$$E[a_k(t_1)a_l(t_2)]^{\mathrm{d}} = \frac{1}{2}\sum_{i=1}^{n} \sigma_i^2 \cos[\beta(d_{kl},\omega_i)] \exp\left[-\frac{1}{2}\alpha^2(d_{kl},\omega_i)\right] \cos[\omega_i(t_2 - t_1)] \tag{18}$$

Taking the Fourier transform, the cross-power spectral density is

$$G_{a_k a_l}(\omega)^{\mathrm{d}} = \frac{1}{4}\sum_{i=1}^{n} \sigma_i^2 \cos[\beta(d_{kl},\omega_i)] \exp\left[-\frac{1}{2}\alpha^2(d_{kl},\omega_i)\right][\delta(\omega_i - \omega) + \delta(\omega_i + \omega)] \tag{19}$$

Using the definition in equation (7) for the power spectral densities of $a_k(t)^{\mathrm{d}}$ and $a_l(t)^{\mathrm{d}}$ and converting the discrete forms of these densities into continuous forms, the preceding equation can be written as

$$G_{a_k a_l}(\omega) = \cos[\beta(d_{kl},\omega)] \exp[-\tfrac{1}{2}\alpha^2(d_{kl},\omega)]\sqrt{G_{a_k a_k}(\omega) G_{a_l a_l}(\omega)} \tag{20}$$

Using the definition in equation (1), this yields

$$\gamma_{kl}(\omega)^{\text{incoherence}} = \cos[\beta(d_{kl},\omega)] \exp[-\tfrac{1}{2}\alpha^2(d_{kl},\omega)] \tag{21}$$

This is the component of the coherency function that characterizes the incoherence effect.

It is seen that the coherency function due to the incoherence effect alone is a real-valued, non-negative function of the inter-station distance d_{kl} and frequency ω. It decays from a value of unity at $d_{kl} = 0$ or $\omega = 0$,

to a value of zero for $d_{kl} \rightarrow \infty$ or $\omega \rightarrow \infty$. Furthermore, using equations (14) and (15), it is easy to see that the derivative of the coherency function with respect to d_{kl} or ω and evaluated at $d_{kl} = 0$ or $\omega = 0$ must be identically zero, provided the derivatives of $\beta(d_{kl}, \omega)$ and $\alpha(d_{kl}, \omega)$ are bounded. This means that the coherency function must have a zero slope at $d_{kl} = 0$ or $\omega = 0$. This behaviour is indeed consistent with empirically determined models of the coherency functions.[2,3,8]

Due to the completely random nature of the incoherence effect, unfortunately it is not possible to go any further in determining the specific forms of the functions $\alpha(d_{kl}, \omega)$ and $\beta(d_{kl}, \omega)$. An appropriate approach is to select models for these functions, e.g. polynomials in d_{kl} and ω, that satisfy the conditions in equations (14)–(17) and use statistical inference to determine the model parameters. It is of course possible to formulate an alternative form of equation (21). For example, the cosine term may be written as $\exp\{\ln(\cos[\beta(d_{kl}, \omega)])\}$ and the two exponents may be combined. However, the form in equation (21) is preferred since it shows the two distinct effects arising from the variabilities in the wave amplitudes and phases separately and conveniently in terms of dimensionless angles.

Considering the propagation of shear waves in a random medium, Luco and Wong[9] derived the lagged coherency function $|\gamma_{kl}(\omega)| = \exp[-\frac{1}{2}(\alpha d_{kl}\omega)^2]$, where α is a constant. More recently, Zerva and Harada[10] proposed a site-specific analytic expression for the constant α on the basis of modelling the soil as a horizontally layered medium with stochastic properties. This model appears to be consistent with the second term in equation (21), which arises from the random variations in the phase angles. However, it is not clear that this model accounts for the random variation in wave amplitudes, which is represented by the first term in equation (21). It is worth noting that the form of the lagged coherency function that Abrahamson *et al.*[3] have used for their empirical studies is markedly different from the theoretically derived form in equation (21).

THE WAVE PASSAGE EFFECT

Consider plane waves arriving at stations k and l with incidence angle ψ (angle with the normal to the ground surface). To account for the delay in the arrival of the wave at station l, we rewrite $a_l(t)^{\mathrm{d}}$ in equation (11) in the form

$$a_l(t)^{\mathrm{d}} = \sum_{i=1}^{n} (p_{kl,i}A_i + q_{kl,i}B_i)\cos[\omega_i(t - \tau_{kl,i}) + \phi_i + \varepsilon_{kl,i}] \tag{22}$$

where $\tau_{kl,i}$ is the lag time for the arrival of the ith component of the wave at station l relative to its arrival at station k. Allowing for dispersion of the waves, we let $v(\omega_i)$ denote the velocity of the ith wave component. One may write,

$$\tau_{kl,i} = \frac{d_{kl}^{\mathrm{L}} \sin\psi}{v(\omega_i)} = \frac{d_{kl}^{\mathrm{L}}}{v_{\mathrm{app}}(\omega_i)} \tag{23}$$

where d_{kl}^{L} denotes the projection of the inter-station distance along the direction of propagation of waves on the ground surface, and $v_{\mathrm{app}}(\omega_i) = v(\omega_i)/\sin\psi$ denotes the apparent wave velocity. Note that the representation in equation (22) includes the effect of incoherence between the two stations.

Following steps similar to those leading to equation (21), we determine the cross-correlation function between the processes $a_k(t)^{\mathrm{d}}$ and $a_l(t)^{\mathrm{d}}$ and take its Fourier transform to determine the cross-power spectral density. After transforming to the continuous form and dividing by the square roots of the power spectral densities of the processes $a_k(t)$ and $a_l(t)$, we find, in accordance with the definition in equation (1),

$$\gamma_{kl}(\omega)^{\text{incoherence + wave passage}} = \cos[\beta(d_{kl}, \omega)]\exp\left[-\frac{1}{2}\alpha^2(d_{kl}, \omega)\right]\exp\left[-\mathrm{i}\frac{\omega d_{kl}^{\mathrm{L}}}{v_{\mathrm{app}}(\omega)}\right] \tag{24}$$

This is the coherency function that includes the effects of incoherence and wave passage. Comparing with equation (21), it is clear that the component of the coherency function for the wave-passage effect

alone is

$$\gamma_{kl}(\omega)^{\text{wave passage}} = \exp\left[-\mathrm{i}\frac{\omega d_{kl}^{\mathrm{L}}}{v_{\mathrm{app}}(\omega)}\right] \tag{25}$$

The preceding formulation allows for dispersion of the seismic waves caused by frequency dependence of the wave velocities. For plane waves in an infinite elastic medium, the wave velocities (for dilatational and shear waves) are independent of frequency and the above formula simplifies to

$$\gamma_{kl}(\omega)^{\text{wave passage}} = \exp\left[-\mathrm{i}\frac{\omega d_{kl}^{\mathrm{L}}}{v_{\mathrm{app}}}\right] \tag{26}$$

where v_{app} is a constant. This form of the coherency function for wave passage has been used by a number of investigators.[7,9,11] Equation (25) generalizes this form to account for the wave dispersion effect, which would be important if the ground medium is not considered as linear elastic. One would normally expect $v_{\mathrm{app}}(\omega)$ to be a slow but increasing function of ω.

By comparison of equations (2) and (25), it is clear that the wave-passage effect, including the effect of wave dispersion, is a deterministic shift in the phasing of the ground motions at the two stations. The frequency-dependent phase angle is

$$\theta_{kl}(\omega)^{\text{wave passage}} = -\frac{\omega d_{kl}^{\mathrm{L}}}{v_{\mathrm{app}}(\omega)} \tag{27}$$

The phase angle is proportional to the frequency and distance between the two stations, and inversely proportional to the wave velocity. The above analysis also shows that the coherency function describing the combined incoherence and wave-passage effects is the product of the coherency functions for the two effects alone.

In closing this section, it is useful to reiterate the basic assumption in deriving the above results, namely that of plane waves arriving at a single incidence angle at the two stations. In reality, waves may arrive at each station from a range of angles (see, e.g. Reference 12). A closed form analytical model for such a complex situation probably cannot be derived. The simple model derived here is useful when there is a predominant angle at which waves arrive at the two stations.

THE ATTENUATION EFFECT

Attenuation due to geometric spreading, material damping and wave scattering decreases the amplitude of each wave component. In general, the decrease in the amplitude is a function of frequency (higher frequency waves tend to attenuate faster) and distance from the earthquake source. To account for these effects, we write the discrete representations of $a_k(t)$ and $a_l(t)$ in the form

$$a_k(t)^{\mathrm{d}} = \sum_{i=1}^{n} A_i f_k(\omega_i, r_k)\cos(\omega_i t + \phi_i) \tag{28}$$

$$a_l(t)^{\mathrm{d}} = \sum_{i=1}^{n} (p_{kl,i}A_i + q_{kl,i}B_i) f_l(\omega_i, r_l)\cos[\omega_i(t - \tau_{kl,i}) + \phi_i + \varepsilon_{kl,i}] \tag{29}$$

in which $0 \leqslant f_m(\omega_i, r_m) \leqslant 1$ for $m = k, l$ are decreasing functions of the frequency ω_i and the source-to-station distance r_m, representing the scaled attenuation law for the region. We note that these functions are not to represent the variations in wave amplitudes caused by scattering and site response effects, which are considered separately (the former through the random wave amplitudes, and the latter in the following section). Rather, these functions represent the average law by which the wave amplitudes decay with distance.

We first consider the case where the functions $f_k(\omega, r_k)$ and $f_l(\omega, r_l)$ are deterministic. The auto- and cross-power spectral densities of the above processes are again determined by first finding the

auto- and cross-correlation functions and subsequently taking Fourier transforms. The results, written in the continuous form, are:

$$G_{a_k a_k}(\omega) = G_{aa}(\omega) f_k^2(\omega, r_k) \tag{30}$$

$$G_{a_l a_l}(\omega) = G_{aa}(\omega) f_l^2(\omega, r_l) \tag{31}$$

$$G_{a_k a_l}(\omega) = G_{aa}(\omega) \cos[\beta(d_{kl}, \omega)] \exp\left[-\frac{1}{2}\alpha^2(d_{kl}, \omega)\right] \exp\left[-\mathrm{i}\,\frac{\omega d_{kl}^{\mathrm{L}}}{v_{\mathrm{app}}(\omega)}\right] f_k(\omega, r_k) f_l(\omega, r_l) \tag{32}$$

where $G_{aa}(\omega)$ denotes the common power spectral density of the processes $a_k(t)$ and $a_l(t)$ in the absence of the attenuation effect. Using these relations in the definition of the coherency function in equation (1), we find that the functions $f_k(\omega, r_k)$ and $f_l(\omega, r_l)$ cancel out and the coherency function remains the same as the one in equation (24). Hence, while deterministic attenuation affects both the amplitude and frequency content of the motion at each station [(as can be seen in the power spectral densities in equations (30) and (31)] and, hence, is a source of spatial variability of the ground motion, it has no influence on the coherency function.

Now suppose the functions $f_k(\omega, r_k)$ and $f_l(\omega, r_l)$ are random and statistically independent of the wave amplitudes and phases. From equations (30)–(32) and (24), the component of the coherency function that arises from the attenuation effect alone takes the form

$$\gamma_{kl}(\omega)^{\text{attenuation}} = \frac{E[f_k(\omega, r_k) f_l(\omega, r_l)]}{\sqrt{E[f_k^2(\omega, r_k) f_l^2(\omega, r_l)]}} \tag{33}$$

which, based on Schwarz's inequality, has a value less than or equal to unity. To examine the significance of this term, consider the extreme case where $f_k(\omega, r_k)$ and $f_l(\omega, r_l)$ are statistically independent. (This case might be relevant when the two stations are very far apart.) The above expression then reduces to $1/\sqrt{(1+\delta_k^2)(1+\delta_l^2)}$, where δ_k and δ_l are the respective coefficients of variation. This expression has a value close to unity even for relatively large coefficients of variation, e.g. for $\delta_k = \delta_l = 0{\cdot}3$ the expression yields 0·92. Any correlation between the two functions, which invariably would be positive, will tend to make the result even closer to unity. Hence, we conclude that attenuation has little influence on the coherency function.

THE SITE RESPONSE EFFECT

Suppose stations k and l are situated on sites with different local soil profiles. Even if the motions at the bedrock level underneath the two stations were identical, the surface motions would be different due to the variation in the filtering effects of the two soil columns. In order to develop a simple model of the coherency function that characterizes this effect, we consider a one-dimensional shear wave propagating vertically in the local soil medium. We also assume that the soil column at each station can be represented as a linear, or linearized, system with known properties.

Let $H_m(\omega)$ denote the frequency-response function of the soil column at station m, $m = k, l$. By definition, this function represents the amplitude of a harmonic motion at the surface of the ground caused by a harmonic motion of the form $\exp(\mathrm{i}\omega t)$ at the bedrock level. Using elementary notions of stationary random vibration theory, the power spectral density of the surface motion at each station is obtained in terms of the power spectral density of the corresponding bedrock motion as

$$G_{a_m a_m}(\omega)^{\text{surface}} = |H_m(\omega)|^2 G_{a_m a_m}(\omega)^{\text{bedrock}}, \quad m = k, l \tag{34}$$

Furthermore, the cross-power spectral density of the surface motions at stations k and l is

$$G_{a_k a_l}(\omega)^{\text{surface}} = H_k(\omega) H_l(-\omega) G_{a_k a_l}(\omega)^{\text{bedrock}} \tag{35}$$

where $G_{a_k a_l}(\omega)^{\text{bedrock}}$ denotes the cross-power spectral density of the two bedrock motions. The latter can be written in the form

$$G_{a_k a_l}(\omega)^{\text{bedrock}} = \gamma_{kl}(\omega)^{\text{bedrock}} \sqrt{G_{a_k a_k}(\omega)^{\text{bedrock}} G_{a_l a_l}(\omega)^{\text{bedrock}}} \tag{36}$$

in which $\gamma_{kl}(\omega)^{\text{bedrock}}$ denotes the coherency function for the two bedrock motions. We assume $\gamma_{kl}(\omega)^{\text{bedrock}}$ accounts for all effects of spatial variability except for the effect of varying site response. Dividing equation (35) by the square roots of equation (34) for $m = k$ and l, and using the identity in equation (36), we obtain the coherency function of the surface motions:

$$\gamma_{kl}(\omega)^{\text{surface}} = \gamma_{kl}(\omega)^{\text{bedrock}} \exp\{\mathrm{i}[\theta_k(\omega) - \theta_l(\omega)]\} \tag{37}$$

where $\theta_m(\omega) = \tan^{-1}\{\mathrm{Im}[H_m(\omega)]/\mathrm{Re}[H_m(\omega)]\}$, $m = k, l$. If the two bedrock motions are assumed to be identical, $\gamma_{kl}(\omega)^{\text{bedrock}} = 1$ and the coherency function due to the site-response effect alone is obtained

$$\gamma_{kl}(\omega)^{\text{site response}} = \exp\{\mathrm{i}[\theta_k(\omega) - \theta_l(\omega)]\} \tag{38}$$

We note that the site-response effect causes a frequency-dependent phase shift of the surface motions at the two stations. The phase angle is given by

$$\begin{aligned} \theta_{kl}(\omega)^{\text{site response}} &= \theta_k(\omega) - \theta_l(\omega) \\ &= \tan^{-1} \frac{\mathrm{Im}[H_k(\omega) H_l(-\omega)]}{\mathrm{Re}[H_k(\omega) H_l(-\omega)]} \end{aligned} \tag{39}$$

Compared to the wave-passage effect, equation (27), one would expect a more complicated dependence of the phase angle on the frequency in this case. It is important to note that the phase shift is completely defined in terms of the two frequency-response functions, which are solely dependent on the properties of the two soil columns. In particular, it does not depend on the distance between the two stations, or the power spectral densities of the bedrock motions

Several methods can be used to determine the frequency-response function of a soil column. One would be to develop an equivalent linear model (e.g. using a program such as SHAKE by Schnabel et al.[13]) and evaluate its steady-state response to a complex harmonic motion. Alternatively, one may determine the response of the soil column to a unit impulse, and then obtain the frequency-response function as its Fourier transform. If earthquake recordings on the surface and bedrock levels (or a nearby rock outcropping) are available, an approximation of the frequency-response function can be obtained by dividing the cross-power spectral density of the two motions by the power spectral density of the bedrock motion. These and other methods are currently under investigation.

It is important to note that the site-response component of the coherency function described above does not account for the effect of incoherency resulting from scattering of waves within the two soil columns. This is obvious by noting that $\gamma_{kl}(\omega)^{\text{site response}} = 1$ if the two stations have identical effective properties, even if the soil medium is randomly heterogeneous. This contribution should be included in the incoherency term, $\gamma_{kl}(\omega)^{\text{incoherence}}$.

COMPOSITE MODEL FOR THE COHERENCY FUNCTION

Combining the components of the coherency function in equations (21), (25) and (38), and ignoring the attenuation effect, a composite model of the coherency function is

$$\begin{aligned} \gamma_{kl}(\omega) &= \gamma_{kl}(\omega)^{\text{incoherence}} \cdot \gamma_{kl}(\omega)^{\text{wave passage}} \cdot \gamma_{kl}(\omega)^{\text{site response}} \\ &= \cos[\beta(d_{kl}, \omega)] \exp[-\tfrac{1}{2}\alpha^2(d_{kl}, \omega)] \exp\{\mathrm{i}[\theta_{kl}(\omega)^{\text{wave passage}} + \theta_{kl}(\omega)^{\text{site response}}]\} \end{aligned} \tag{40}$$

where the phase angles $\theta_{kl}(\omega)^{\text{wave passage}}$ and $\theta_{kl}(\omega)^{\text{site response}}$ are given in equations (27) and (39), respectively.

The following procedure is suggested for determining a site-specific coherency function using the above model:

(1) Make an estimate of the lagged coherency function, i.e. the modules $|\gamma_{kl}(\omega)|$, by using recordings of a dense array having features similar to the site under study. This represents the $\gamma_{kl}(\omega)^{\text{incoherence}}$ component. Alternatively, the contribution from the extended source effect can be estimated by simulating time histories as suggested by Somerville et al.,[5] to which the incoherence effect due to scattering must be added. The expression in equation (21) may be used as a model for the regression analysis.
(2) Make an estimate of the apparent wave velocity by considering the average wave velocity over the region and the source-to-site geometry. For deep soil layers, the incidence angle considered should be that at the bedrock level. Alternatively, the lag time $\tau_{kl,i}$ can be estimated empirically by examining dense array recordings in the manner described, e.g. by Hao *et al.*[8] If desired and justifiable, a frequency dependent wave velocity may be used. These estimates together with equation (27) define the phase angle function $\theta_{kl}(\omega)^{\text{wave passage}}$ of the coherency function.
(3) Determine the site-response component of the coherency function by generating the frequency-response function at each station using one of the methods described earlier. Note that only the effective properties of the soil layer at each station are needed for this purpose. Equation (39) then provides the phase angle function $\theta_{kl}(\omega)^{\text{site response}}$ for each pair of stations.

RELATIVE IMPORTANCE OF WAVE PASSAGE AND SITE RESPONSE EFFECTS

It is seen in equation (40) that the effects of wave passage and site response are both phase shifts and are additive in nature. It is worthwhile to compare these two effects and determine their relative importance.

To describe the soil layers at station k and l, we adopt the frequency-response function

$$H_m(\omega) = \frac{\omega_m^2 + 2i\zeta_m\omega_m\omega}{\omega_m^2 - \omega^2 + 2i\zeta_m\omega_m\omega}, \quad m = k, l \tag{41}$$

in which ω_m and ζ_m are parameters. This model idealizes the soil layer as a single-degree-of-freedom oscillator of frequency ω_m and damping ratio ζ_m. (This model is the basis of the so called Kanai–Tajimi power spectral density of surface ground motions. The K–T power spectral density is obtained by subjecting the oscillator to a white noise base acceleration; see Reference 6.) In Figure 1, the transfer function $|H_m(\omega)|^2$ is plotted for three sets of parameter values that idealize sites with soft (or deep), medium, and stiff (or shallow) soil properties. The transfer function for the soft site is narrow and has a sharp peak around 0·5 Hz frequency. The transfer function for the medium site is broader and has a less sharp peak around 1·4 Hz frequency. Finally, the transfer function for the stiff site is the broadest and has a shallow peak around 2·5 Hz. It is believed that these transfer functions are reasonably realistic characterizations of sites with soft, medium, and stiff soil properties.

Using equations (39) and (41), the phase angle $\theta_{kl}(\omega)^{\text{site response}}$ is plotted as solid lines in Figure 2 against the frequency for three different pairs of sites with soft and medium, medium and stiff, and soft and stiff properties. Note that for identical sites, the phase angle due to the site-response effect is equal to zero. As expected, the phase angle is largest for the pair of sites with the most disparate properties.

For a constant apparent wave velocity, the phase angle for the wave-passage effect, equation (27), is a linear function of ω. This is plotted in Figure 2 as dotted lines for $d_{kl}^{\text{L}}/v_{\text{app}} = 0{\cdot}02$ s and $d_{kl}^{\text{L}}/v_{\text{app}} = 0{\cdot}2$ s. These values represent the range of $d_{kl}^{\text{L}}/v_{\text{app}}$ typically encountered in practice. In particular, the value $d_{kl}^{\text{L}}/v_{\text{app}} = 0{\cdot}2$ s corresponds to, e.g. $d_{kl}^{\text{L}} = 200$ m and $v_{\text{app}} = 1000$ m/s.

It is seen in Figure 2 that the phase angle for the site-response effect for pairs of sites with soft and medium or soft and stiff properties is larger than the phase angle for the wave-passage effect for frequencies in the range 0·5–1·5 Hz. This analysis reveals that the relative significance of the wave-passage and site-response effects depends on the fundamental frequency of the structure. For multiply supported structures situated in

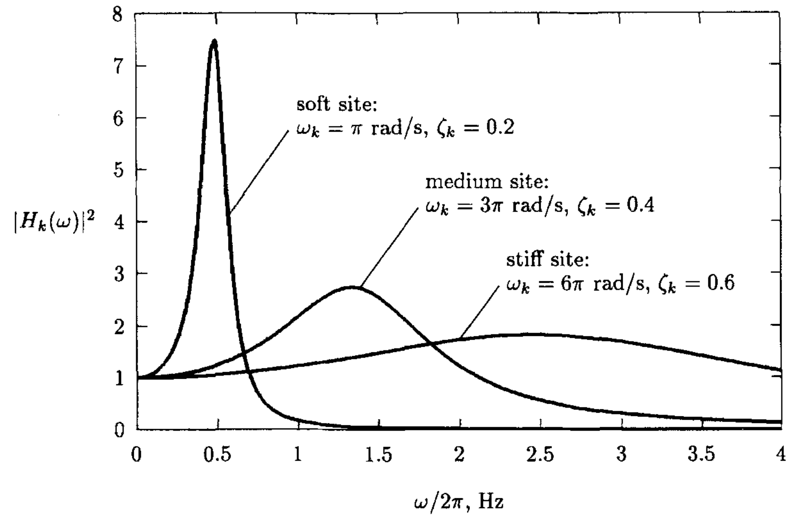

Figure 1. Frequency-response functions for idealized sites

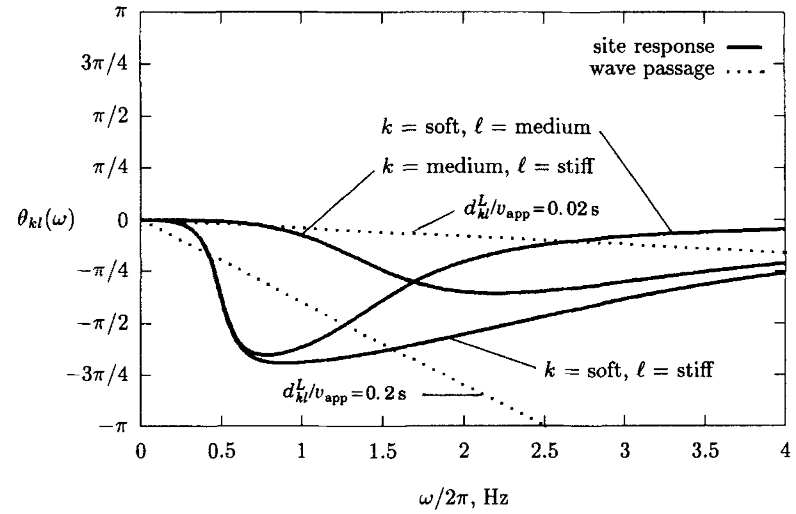

Figure 2. Coherency phase-angle for site response and wave-passage effects

regions with rapidly varying local soil properties, the effect of site response can be more significant if the structure has important modes within the range 0·5–1·5 Hz. Many bridges and viaducts with short or moderate span lengths fall in this category. For more flexible structures, such as long-span suspension bridges, the wave-passage effect is expected to be more significant.

CONCLUSIONS

A theoretical model for the coherency function describing the spatial variability of earthquake ground motions is developed. The model is derived from basic principles of random process theory and a set of simplifying assumptions regarding the propagation of seismic waves through the ground medium. The model is composed of three components characterizing the distinct effects that give rise to the spatial variability,

namely, the incoherence effect, the wave-passage effect, and the site-response effect. Attenuation is found to have little influence on the coherency function.

The incoherence effect, which is due to scattering of waves in the ground medium and their differential superpositioning when arriving from an extended source, characterizes the loss of statistical dependence between the motions at two stations. This component of the coherency function is a real-valued, non-negative, decaying function of frequency and inter-station distance. It is shown to be formed of two sub-components: one representing the effect of random phase variations, and one representing the effect of random amplitude variations between the wave components at two stations.

The wave-passage component of the coherency function characterizes the spatial variability of the ground motion arising from the difference in the arrival times of waves at separate stations. This component is represented in terms of a phase angle expressed as a function of frequency, inter-station distance, and the apparent velocity of seismic waves.

The site-response component of the coherency function characterizes the spatial variability of the ground motion arising from the difference in the local soil conditions at two stations. This component is also represented in terms of a phase angle function. However, for this effect, the phase angle is dependent on the properties of the local soil profile at each station; it is not a function of the inter-station distance.

Comparison between the wave-passage and site-response effects reveals that the site-response effect can be more significant for structures situated in regions with rapidly changing local soil conditions and which have important natural modes with frequencies in the range 0·5–1·5 Hz. Bridges and viaducts with short or medium length spans may fall in this category. For long-span, flexible structures, such as suspension bridges, with frequencies lower than 0·5 Hz, the wave-passage effect is expected to be more significant.

In closing, it is noted that any mathematical model is imperfect, and the model developed in this paper is no exception. The characterization of the ground motion as a stationary process, the assumption of plane waves made to describe the wave-passage effect, the idealization of waves propagating vertically through the local soil columns, and the linear characterization of the soil behaviour are all simple idealizations of complex reality. Nevertheless, the model developed in this paper provides a mathematical framework that may allow better calibration with recorded data, as well as specification of design motions for regions or geologic settings where no array recordings are available. Clearly, there is need for verification of the model against recorded data, and work along this line is currently being pursued.

APPENDIX I

Consider

$$z = \phi + \varepsilon \tag{42}$$

where ϕ is a random angle distributed uniformly in the interval $(0, 2\pi)$, and ε is a random angle defined over the interval $(-\infty, \infty)$ and independent of ϕ. Using rules of probability transformation, the probability density function of z is

$$f(z) = \frac{1}{2\pi}\left[F_\varepsilon(z) - F_\varepsilon(z - 2\pi)\right] \quad -\infty < z < \infty \tag{43}$$

where $F_\varepsilon(\cdot)$ denotes the cumulative distribution function of ε. Let $\bar{z}$ denote an angle within the interval $(0, 2\pi)$ obtained by substracting from z positive or negative multiples of 2π. The probability density of $\bar{z}$ is obtained by adding the densities of $z = \bar{z} + 2\pi k$ for all integers k, i.e.

$$f(\bar{z}) = \frac{1}{2\pi}\lim_{j\to\infty}\left[\sum_{k=-j}^{k=j} F_\varepsilon(\bar{z} + 2\pi k) - F_\varepsilon(\bar{z} - 2\pi + 2\pi k)\right]$$

$$= \frac{1}{2\pi} \lim_{j \to \infty} [F_\varepsilon(\bar{z} + 2\pi j) - F_\varepsilon(\bar{z} - 2\pi - 2\pi j)]$$

$$= \frac{1}{2\pi}, \quad 0 \leqslant \bar{z} \leqslant 2\pi \tag{44}$$

This proves that the wrapped angle $\bar{z}$ is uniformly distributed over the interval $(0, 2\pi)$.

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